AU-6858

B. Sc.-I, Mathematics-I (Algebra)

Model Solution / Suggestive Answers

1. (i) (a) For unique solution: $\rho(A)$ = number of variables

(b) For infinitely many solutions: $\rho(A) <$ number of variables

(ii) $(\mathbb{N}, +)$ is a semigp. but not a group.

(iii) $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

Composition table

$\oplus_4$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

(iv) The smallest positive integer n such that $a^n = e$ is called order of 'a' in a group G. For $G = \{1, -1, i, -i\}$

$o(1) = 1$, $o(-1) = 2$, $o(i) = o(-i) = 4$.

(v) Klien's 4-group $V_4 = \{e, a, b, c \mid a^2 = b^2 = c^2 = e, a \cdot b = c, b \cdot c = a, c \cdot a = b\}$ is an abelian group but not cyclic as order of each non-identity element is 2.

(vi) Number of generators of G $= \phi(15)$

$= \phi(3 \times 5)$

$= \phi(3) \cdot \phi(5)$

$= (3^1 - 3^0) \cdot (5^1 - 5^0)$

$= (3-1)(5-1)$

$= 8$

(vii) G, being a prime order group, is cyclic and hence abelian. $H \trianglelefteq G$ as every subgroup of an abelian group is normal.

(viii) $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \Rightarrow f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$,

$g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

2. Determinant of the given matrix is non-zero, hence its inverse exists. Now

$$M = \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right], \text{ by } R_1 \to \tfrac{1}{2} R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right], \text{ by } R_2 \to R_2 - 5R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{5}{2} & -1 & 1 \end{array}\right], \text{ by } R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right], \text{ by } R_3 \to 2R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right], \text{ by } R_1 \to R_1 + \tfrac{1}{2} R_3 \text{ and } R_2 \to R_2 - \tfrac{5}{2} R_3$$

Hence inverse of the given matrix is $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$. □

3. The augmented matrix

$$M = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 12 \\ 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & k \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 2 & 1 & -1 & 12 \\ 0 & 3 & 3 & k \end{array}\right]$$

$$\overset{R_2 \to R_2 - 2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & 18 \\ 0 & 3 & 3 & k \end{array}\right] \overset{R_2 \to \frac{1}{3} R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 1 & 6 \\ 0 & 3 & 3 & k \end{array}\right]$$

$$\overset{R_3 \to R_3 - 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & k-18 \end{array}\right]$$

The last matrix is in row reduced echelon form. For the consistency of the system $\rho(A) = \rho(M) = 2$, i.e. $k - 18 = 0$, i.e. $\boxed{k = 18}$.

Number of l.i. solutions of the system $= n - \rho + 1 = 3 - 2 + 1 = 2$.

Now the reduced system is

$$\begin{bmatrix} x - y - 2z = -3 \\ y + z = 6 \end{bmatrix}.$$

Choosing $z = 0, 1$, we get two l.i. solutions as follows:

x	3	4
y	6	5
z	0	1

$\square$

4. Result follows from induction on n and commutativity of G. $\square$

5. Assume that $H \cup K \leq G$. Suppose that $K \nsubseteq H$. Now, take any $h \in H$ and $k \in K - H$. Then $h, k \in H \cup K$ and hence $h \circ k \in H \cup K$ as $H \cup K$ is a subgroup. This implies $h \circ k \in H$ or $h \circ k \in K$. The former shows $k \in H$ (which is ~~a contradiction~~ impossible as $k \in K - H$) and the latter gives $h \in K$. Hence, we have $H \subseteq K$. Similarly, we can show that if $H \nsubseteq K$, then $K \subseteq H$.

Conversely, assume that either $H \subseteq K$ or $K \subseteq H$. Then $H \cup K = K$ or $H \cup K = H$, in each case $H \cup K \leq G$. $\square$

6. Suppose that $o(a) = n$. Then $a^0 = e, a, a^2, a^3, \ldots, a^{n-1}$ are distinct elements of G. For, if $a^r = a^s$ $(0 \le r < s \le n)$, then $a^{s-r} = e$, ~~Therefore $s - r = 0$~~, i.e. this is a contradiction to $o(a) = n$. Now, we show that the above listed elements are all the elements of G. Take any a^t, for some $t \in \mathbb{Z}$. Dividing t by n, we get a $q (\neq 0)$ and $r \in \mathbb{Z}$ such that

$$t = nq + r,$$

$0 \le r < n$. Now $a^t = a^{nq + r} = (a^n)^q \circ a^r = e \circ a^r = a^r$, which belongs to the above list. Hence $G = \{e, a, a^2, \ldots, a^{n-1}\}$. Thus

$$o(a) = o(G) = n.$$

$\square$

(7) Lagrange's theorem: For any subgroup H of a finite group G, $o(H)$ divides $o(G)$.

Proof: Suppose that Ha_i, $i=1,2,\ldots,k$ are distinct right cosets of H in G. For any $a \in G$, $a \in Ha = Ha_i$, for some $i \in \{1,2,\ldots,k\}$. Therefore every element of G belongs to one of the above listed cosets. ~~Since these cosets are distinct, they are pairwise disjoint.~~ Hence $G = \bigcup_{i=1}^{k} Ha_i$.
Since the above listed cosets are distinct, they are pairwise disjoint. Therefore

$$o(G) = \sum_{i=1}^{k} o(Ha_i)$$
$$= o(H) + o(H) + \cdots + o(H) \quad (k\text{-times})$$
$$= k \cdot o(H)$$

This shows that $o(H)$ divides $o(G)$. $\square$

8. A bijective homomorphism $f: G \longrightarrow G'$ is called an isomorphism.
Suppose that $G = \langle a \rangle$ is an infinite cyclic group. Define a map $f: \mathbb{Z} \longrightarrow G$ by $f(k) = a^k$. This map is a surjective map (it is easy to observe). It is injective as $f(r) = f(s) \Rightarrow a^r = a^s \Rightarrow a^{r-s} = e$ $\Rightarrow r-s=0$ (as G is an infinite cyclic group) $\Rightarrow r=s$. It is a homomorphism as $f(r+s) = a^{r+s} = a^r \circ a^s = f(r) \circ f(s)$. Thus $G \approx \mathbb{Z}$ (keeping in mind that 'being isomorphic to' is an equivalence relation). $\square$

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